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When does the k-hyponormality of a 2-variable weighted shift become subnormality? ☆

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ABSTRACT

In this article we construct a sequence of nontrivial classes of 2-variable weighted shifts $\{\mathcal{G}_k\}_{k=2}^{\infty}$ such that the k-hyponormality of an arbitrary power of a member $W_{(\alpha,\beta)}$ from $\mathcal{G}_k$ is equivalent to its subnormality.

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded linear operators on $\mathcal{H}$. For $S, T \in \mathcal{B}(\mathcal{H})$ let $[S, T] := ST - TS$. We say that an n-tuple $\mathbf{T} = (T_1, \dots, T_n)$ of operators on $\mathcal{H}$ is (jointly) *hyponormal* if the operator matrix

$$[\mathbf{T}^*, \mathbf{T}] := \begin{pmatrix} [T_1^*, T_1] & [T_2^*, T_1] & \cdots & [T_n^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] & \cdots & [T_n^*, T_2] \\ \vdots & \vdots & \ddots & \vdots \\ [T_1^*, T_n] & [T_2^*, T_n] & \cdots & [T_n^*, T_n] \end{pmatrix}$$

is positive on the direct sum of n copies of $\mathcal{H}$ (cf. [1,10]). The n-tuple $\mathbf{T}$ is said to be *normal* if $\mathbf{T}$ is commuting and each T_i is normal, and $\mathbf{T}$ is *subnormal* if $\mathbf{T}$ is the restriction of a normal n-tuple to a common invariant subspace. For $k \geqslant 1$, a commuting pair $\mathbf{T} \equiv (T_1, T_2)$ is said to be *k-hyponormal* if

$$\mathbf{T}(k) := \left(T_1, T_2, T_1^2, T_2T_1, T_2^2, \dots, T_1^k, T_2T_1^{k-1}, \dots, T_2^k\right)$$

is hyponormal, or equivalently

$$\left[\mathbf{T}(k)^*, \mathbf{T}(k)\right] = \left(\left[\left(T_2^q T_1^p\right)^*, T_2^m T_1^n\right]\right)_{\substack{1 \leqslant n+m \leqslant k \\ 1 \leqslant p+q \leqslant k}} \geqslant 0.$$

Clearly, normal $\Rightarrow$ subnormal $\Rightarrow$ k-hyponormal. The Bram–Halmos criterion states that an operator $T \in \mathcal{B}(\mathcal{H})$ is subnormal if and only if the k-tuple $(T, T^2, \dots, T^k)$ is hyponormal for all $k \geqslant 1$.

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For $\alpha \equiv \{\alpha_n\}_{n=0}^{\infty}$ a bounded sequence of positive real numbers (called *weights*), let $W_\alpha : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ be the associated unilateral weighted shift, defined by $W_\alpha e_n := \alpha_n e_{n+1}$ (all $n \geqslant 0$), where $\{e_n\}_{n=0}^{\infty}$ is the canonical orthonormal basis in $\ell^2(\mathbb{Z}_+)$. We let $W_\alpha \equiv \operatorname{shift}(\alpha_0, \alpha_1, \ldots)$ and $U_+ := \operatorname{shift}(1, 1, \ldots)$ (the (unweighted) unilateral shift). For $0 < a \leqslant 1$ we also let $S_a := \operatorname{shift}(a, 1, 1, \ldots)$. The moments of W_α are given as

$$\gamma_k \equiv \gamma_k(W_\alpha) := \left\{ \begin{array}{ll} 1, & \text{if } k = 0 \\ \alpha_0^2 \cdots \alpha_{k-1}^2, & \text{if } k > 0 \end{array} \right\}.$$

It is easy to see that W_α is never normal, and that it is hyponormal if and only if $\alpha_0 \leqslant \alpha_1 \leqslant \cdots$. Similarly, consider double-indexed positive bounded sequences $\alpha_{\mathbf{k}}, \beta_{\mathbf{k}} \in \ell^\infty(\mathbb{Z}_+^2)$, $\mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2 := \mathbb{Z}_+ \times \mathbb{Z}_+$ and let $\ell^2(\mathbb{Z}_+^2)$ be the Hilbert space of square-summable complex sequences indexed by $\mathbb{Z}_+^2$. (Recall that $\ell^2(\mathbb{Z}_+^2)$ is canonically isometrically isomorphic to $\ell^2(\mathbb{Z}_+) \otimes \ell^2(\mathbb{Z}_+)$.) We define the 2-variable weighted shift $W_{(\alpha,\beta)} \equiv (T_1, T_2)$ by

$$T_1 e_{\mathbf{k}} := \alpha_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_1},$$
$$T_2 e_{\mathbf{k}} := \beta_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_2},$$

where $\varepsilon_1 := (1, 0)$ and $\varepsilon_2 := (0, 1)$. Clearly,

$$T_1 T_2 = T_2 T_1 \quad \Leftrightarrow \quad \beta_{\mathbf{k}+\varepsilon_1} \alpha_{\mathbf{k}} = \alpha_{\mathbf{k}+\varepsilon_2} \beta_{\mathbf{k}} \quad \left(\text{all } \mathbf{k} \in \mathbb{Z}_+^2\right). \tag{1.1}$$

In an entirely similar way one can define multivariable weighted shifts. Trivially, a pair of unilateral weighted shifts W_a and W_β gives rise to a 2-variable weighted shift $W_{(\alpha,\beta)}$, if we let $\alpha_{(k_1,k_2)} := \alpha_{k_1}$ and $\beta_{(k_1,k_2)} := \beta_{k_2}$ (all $k_1, k_2 \in \mathbb{Z}_+$). In this case, $W_{(\alpha,\beta)}$ is subnormal (resp. hyponormal) if and only if so are T_1 and T_2; in fact, under the canonical identification of $\ell^2(\mathbb{Z}_+^2)$ with $\ell^2(\mathbb{Z}_+) \otimes \ell^2(\mathbb{Z}_+)$, we have $T_1 \cong I \otimes W_a$ and $T_2 \cong W_\beta \otimes I$, and $W_{(\alpha,\beta)}$ is also doubly commuting. For this reason, we do not focus attention on shifts of this type, and use them only when the above mentioned triviality is desirable or needed. Given $\mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2$, the moments of $W_{(\alpha,\beta)}$ of order $\mathbf{k}$ are

$$\gamma_{\mathbf{k}} \equiv \gamma_{\mathbf{k}}(W_{(\alpha,\beta)}) := \begin{cases} 1, & \text{if } (k_1, k_2) = (0, 0), \\ \alpha_{(0,0)}^2 \cdots \alpha_{(k_1-1,0)}^2, & \text{if } k_1 \geqslant 1 \text{ and } k_2 = 0, \\ \beta_{(0,0)}^2 \cdots \beta_{(0,k_2-1)}^2, & \text{if } k_1 = 0 \text{ and } k_2 \geqslant 1, \\ \alpha_{(0,0)}^2 \cdots \alpha_{(k_1-1,0)}^2 \beta_{(k_1,0)}^2 \cdots \beta_{(k_1,k_2-1)}^2, & \text{if } k_1 \geqslant 1 \text{ and } k_2 \geqslant 1. \end{cases} \tag{1.2}$$

We remark that, due to the commutativity condition (1.1), $\gamma_{\mathbf{k}}$ can be computed using any nondecreasing path from $(0, 0)$ to (k_1, k_2). We now recall a well-known characterization of subnormality for multivariable weighted shifts [20], due to C. Berger (cf. [2, III.8.16]) and independently established by R. Gellar and L.J. Wallen [18] in the single variable case: $W_{(\alpha,\beta)}$ admits a commuting normal extension if and only if there is a probability measure μ (which we call the *Berger measure* of $W_{(\alpha,\beta)}$) defined on the 2-dimensional rectangle $R = [0, a_1] \times [0, a_2]$ (where $a_i := \|T_i\|^2$) such that $\gamma_{\mathbf{k}} = \int_R \mathbf{t}^{\mathbf{k}}\, d\mu(\mathbf{t}) := \int_R t_1^{k_1} t_2^{k_2}\, d\mu(\mathbf{t})$, for all $\mathbf{k} \in \mathbb{Z}_+^2$. Observe that U_+ and S_a are subnormal, with Berger measures δ_1 and $(1 - a^2)\delta_0 + a^2\delta_1$, respectively, where δ_p denotes the point-mass probability measure with support the singleton set $\{p\}$. Single and multivariable weighted shifts have played an important role in the study of the problems of existence of commuting normal extensions (cf. [5–7,9, 13–15,21,26]). They have also played a significant role in the study of cyclicity and reflexivity, in the study of C^*-algebras generated by multiplication operators on Bergman spaces, as fertile ground to test new hypotheses, and as canonical models for theories of dilation and positivity (cf. [12,19,22]). We need some further notation to describe our results. We use $\mathfrak{H}_0$ (resp. $\mathfrak{H}_\infty$) to denote the set of commuting pairs of subnormal operators (resp. subnormal pairs) on Hilbert space. For $k \geqslant 1$, we let $\mathfrak{H}_k$ denote the class of k-hyponormal pairs in $\mathfrak{H}_0$. Clearly, $\mathfrak{H}_\infty \subseteq \cdots \subseteq \mathfrak{H}_k \subseteq \cdots \subseteq \mathfrak{H}_2 \subseteq \mathfrak{H}_1 \subseteq \mathfrak{H}_0$. The main results in [5,13] show that these inclusions are all proper. For an arbitrary 2-variable weighted shift $W_{(\alpha,\beta)}$, we let $\mathcal{M}_i$ (resp. $\mathcal{N}_j$) be the subspace of $\ell^2(\mathbb{Z}_+^2)$ which is spanned by the canonical orthonormal basis associated to indices $\mathbf{k} \equiv (k_1, k_2)$ with $k_1 \geqslant 0$ and $k_2 \geqslant i$ (resp. $k_1 \geqslant j$ and $k_2 \geqslant 0$). We will often write $\mathcal{M}_1$ simply as $\mathcal{M}$ and $\mathcal{N}_1$ as $\mathcal{N}$. The core $c(W_{(\alpha,\beta)})$ of $W_{(\alpha,\beta)}$ is the restriction of $W_{(\alpha,\beta)}$ to the invariant subspace $\mathcal{M} \cap \mathcal{N}$. A 2-variable weighted shift $W_{(\alpha,\beta)}$ is said to be *of tensor form* if it is of the form $(I \otimes W_\alpha, W_\beta \otimes I)$. The class of all 2-variable weighted shifts $W_{(\alpha,\beta)} \in \mathfrak{H}_0$ whose core is of tensor form will be denoted by $\mathcal{TC}$; in symbols, $\mathcal{TC} := \{W_{(\alpha,\beta)} \in \mathfrak{H}_0\colon\ c(W_{(\alpha,\beta)}) \text{ is of tensor form}\}$. Note that if $W_{(\alpha,\beta)} \in \mathcal{TC}$, then $W_{(\alpha,\beta)} \in \mathfrak{H}_\infty$. Given a subnormal 2-variable weighted shift $W_{(\alpha,\beta)}$ with Berger measure μ, we let $W_{\alpha^{(j)}}$ $(j \geqslant 0)$ (resp. $W_{\beta^{(i)}}$ $(i \geqslant 0)$) denote the associated j-th horizontal (resp. i-th vertical) slice of $W_{(\alpha,\beta)}$. Clearly, $W_{\alpha^{(j)}}$ (resp. $W_{\beta^{(i)}}$) is subnormal, and we let ξ_j (resp. η_i) denote its Berger measure. For $k \geqslant 1$ we let $\mathcal{G}_k := \{W_{(\alpha,\beta)} \in \mathcal{A}\colon\ W_{(\alpha,\beta)}|_{\mathcal{M}} \equiv (I \otimes S_a, U_+ \otimes I)$ and $\operatorname{card}(\operatorname{supp} \xi_0) \leqslant (k+1)\}$, where $\mathcal{A} := \{W_{(\alpha,\beta)} \in \mathcal{TC}\colon\ c(W_{(\alpha,\beta)}) \text{ has 1-atomic Berger measure}\}$ and $\operatorname{card}(\operatorname{supp} \xi_0)$ means the cardinality of the support of ξ_0 (cf. see Fig. 1(ii)). For the meaning of set inclusion, we clearly have $\mathcal{G}_1 = \mathcal{S}_1 \subsetneq \mathcal{G}_2 \subsetneq \cdots \subsetneq \mathcal{G}_k \subsetneq \cdots \subsetneq \mathcal{A} \subsetneq \mathcal{TC}$. Observe that a 2-variable weighted shift $W_{(\alpha,\beta)} \in \mathcal{S}_1$ has a core with Berger measure $\delta_{\{1,1\}} = \delta_1 \times \delta_1$. For $k \geqslant 1$, we note that if $W_{(\alpha,\beta)} \in \mathcal{G}_k$, then $W_{(\alpha,\beta)}$ can be fully determined by 3 parameters: the weight $a := \alpha_{(0,1)}$, the weight $y := \beta_{(0,0)}$ and the Berger measure ξ_0 of the 0-th horizontal subnormal slice $\operatorname{shift}(x_0, x_1, x_2, \ldots)$ of $W_{(\alpha,\beta)}$. Thus we can denote a 2-variable weighted shift $W_{(\alpha,\beta)} \in \mathcal{G}_k$ by $\langle a, y, \xi_0\rangle$.

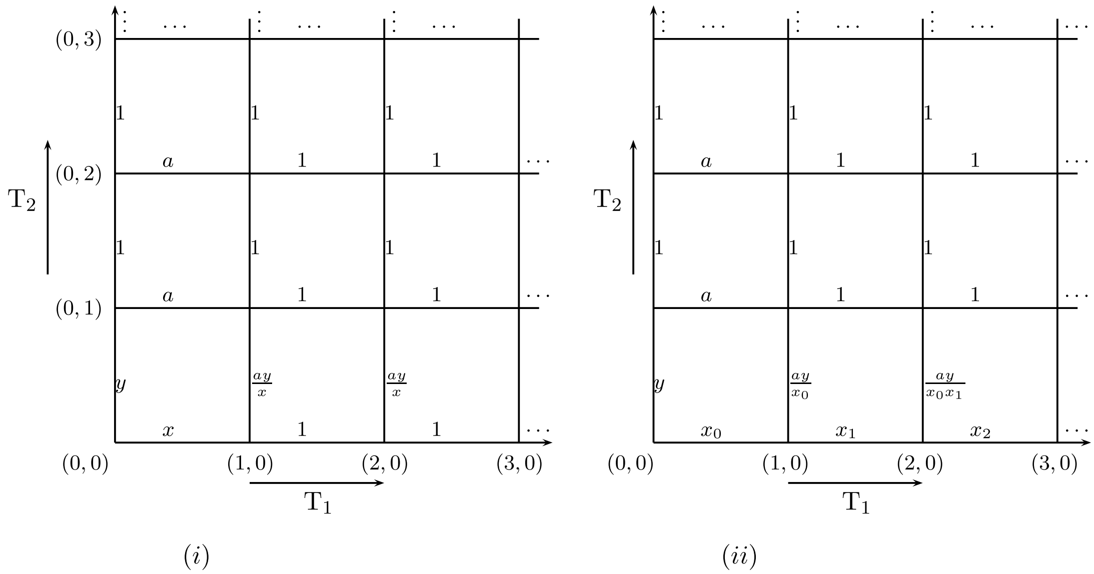

Fig. 1. Weight diagram of a generic 2-variable weighted shift in $\mathcal{S}_1$ and the 2-variable weighted shift in Theorem 2.5, respectively.

2. Main results

For a general operator T on Hilbert space, it is well known that the subnormality of T implies the subnormality of T^h $(h \geqslant 2)$. The converse implication, however, is false; in fact, the subnormality of all powers T^h $(h \geqslant 2)$ does not necessarily imply the subnormality of T, even if $T \equiv W_\alpha$ is a unilateral weighted shift [24, p. 378]. Thus, it is natural to ask when the subnormality of all powers W_α^h $(h \geqslant 2)$ does imply the subnormality of W_α. More general, we might ask when the k-hyponormality $(k \geqslant 1)$ of all powers W_α^h $(h \geqslant 2)$ does imply the subnormality of W_α. For $k \geqslant 2$, we let $W_\alpha \equiv \operatorname{shift}(a, b, 1, 1, \ldots)$ where $0 < a < b < 1$. Then W_α^h $(h \geqslant 2)$ is subnormal, but W_α is not k-hyponormal $(k \geqslant 2)$, because the k-hyponormality of W_α $(k \geqslant 2) \Leftrightarrow b = 1$. For $k = 1$, we consider $W_\alpha \equiv \operatorname{shift}(1, 1 - x, y, y, \ldots)$ where $0 < x < 1 < y$. Then a simple calculation shows that W_α^h $(h \geqslant 2)$ is subnormal, but W_α is not hyponormal. In the multivariable case, we can consider these analogous results. The standard assumption on a pair $\mathbf{T} \equiv (T_1, T_2)$ is that each component T_i is subnormal $(i = 1, 2)$. With this in mind, the analogous questions are highly nontrivial. In [8,17], we identified a large nontrivial class $\mathcal{S}_1$ (cf. see Fig. 1(i)) of 2-variable weighted shifts for which the 2-hyponormality of an arbitrary power of the initial pair is equivalent to subnormality of the initial pair. Thus, it is natural to consider

Problem 2.1. (See [17, Problem 6.8].) Is $\mathcal{S}_1$ the largest class in $\mathcal{A}$ for which the implication

$$W_{(\alpha,\beta)}^{(h_0,\ell_0)} := \left(T_1^{h_0}, T_2^{\ell_0}\right) \in \bigoplus \mathfrak{H}_2 \quad \text{for some } h_0, \ell_0 \geqslant 1 \quad \Rightarrow \quad W_{(\alpha,\beta)} \in \mathfrak{H}_\infty$$

holds?

In this paper we give a concrete answer for Problem 2.1 above and build a class $\mathcal{G}_k$ $(k \geqslant 2)$ in $\mathcal{A}$ such that if $W_{(\alpha,\beta)} \in \mathcal{G}_k$ with $\operatorname{card}(\operatorname{supp} \xi_0) = k + 1$, then for some $h_0, \ell_0 \geqslant 1$

$$W_{(\alpha,\beta)}^{(h_0,\ell_0)} \in \bigoplus \mathfrak{H}_k \quad \Leftrightarrow \quad W_{(\alpha,\beta)} \in \mathfrak{H}_\infty.$$

For this, we first recall that, in one variable, the n-th power of a weighted shift is unitarily equivalent to the direct sum of n weighted shifts. Something similar happens in two variables, as we will see in the proof of Theorem 2.5 below. We let $\mathcal{H} \equiv \ell^2(\mathbb{Z}_+) = \bigvee_{j=0}^{\infty}\{e_j\}$. Given integers i and h $(h \geqslant 1,\ 0 \leqslant i \leqslant h - 1)$, define $\mathcal{H}_i := \bigvee_{j=0}^{\infty}\{e_{hj+i}\}$; clearly, $\mathcal{H} = \bigoplus_{i=0}^{h-1} \mathcal{H}_i$. Following the notation in [11], for a weight sequence $\alpha \equiv \{\alpha_n\}_{n=0}^{\infty}$ we let

$$W_{\alpha(h:i)} := \operatorname{shift}\left(\prod_{n=0}^{h-1} \alpha_{hj+i+n}\right)_{j=0}^{\infty}; \tag{2.1}$$

that is, $W_{\alpha(h:i)}$ denotes the sequence of products of weights in adjacent packets of size h, beginning with $\alpha_i \cdots \alpha_{i+h-1}$. For example, given a weighted shift $W_\alpha \equiv \operatorname{shift}(\alpha_0, \alpha_1, \ldots)$, we have $W_{\alpha(2:0)} = \operatorname{shift}(\alpha_0\alpha_1, \alpha_2\alpha_3, \ldots)$, $W_{\alpha(2:1)} = \operatorname{shift}(\alpha_1\alpha_2, \alpha_3\alpha_4, \ldots)$ and $W_{\alpha(3:2)} = \operatorname{shift}(\alpha_2\alpha_3\alpha_4, \alpha_5\alpha_6\alpha_7, \ldots)$. For $h \geqslant 1$, $0 \leqslant i \leqslant h - 1$, we note that $W_{\alpha(h:i)}$ is unitarily equivalent to $W_\alpha^h|_{\mathcal{H}_i}$. Therefore, W_α^h is unitarily equivalent to $\bigoplus_{i=0}^{h-1} W_{\alpha(h:i)}$. Consequently, if W_α is subnormal with the

Berger measure μ, then $W_{\alpha(h,i)}$ is subnormal with the Berger measure μ_i, where

$$d\mu_0(s) = d\mu\left(s^{\frac{1}{h}}\right) \quad \text{and} \quad d\mu_i(s) = \frac{s^{\frac{1}{h}}}{\gamma_i} d\mu\left(s^{\frac{1}{h}}\right) \quad \text{for } 1 \leqslant i \leqslant h-1. \tag{2.2}$$

Furthermore, we have

Lemma 2.2. *(See [11, Corollary 2.8].)*

(i) *Let $k \geqslant 1$. Then W_α^h is k-hyponormal $\Leftrightarrow$ $W_{\alpha(h:i)}$ is k-hyponormal for $0 \leqslant i \leqslant h-1$.*
(ii) *W_α^h is subnormal $\Leftrightarrow$ $W_{\alpha(h:i)}$ is subnormal for $0 \leqslant i \leqslant h-1$.*

We now introduce a key family of examples for our main results. For $k \geqslant 2$, $0 < a_i$ $(1 \leqslant i \leqslant k-1)$, $0 \leqslant c_0, c_1, \ldots, c_k \leqslant 1$ (with $\sum_{i=0}^k c_i = 1$), we let $W_x \equiv \operatorname{shift}(x_0, x_1, \ldots)$ be given by

$$x_n := \begin{cases} \sqrt{\sum_{i=1}^{k-1} c_i a_i + c_k}, & \text{if } n = 0, \\ \sqrt{\frac{\sum_{i=1}^{k-1} c_i a_i^{n+1} + c_k}{\sum_{i=1}^{k-1} c_i a_i^n + c_k}}, & \text{if } n \geqslant 1. \end{cases} \tag{2.3}$$

We now consider

$$\xi_0 := c_0\delta_0 + \sum_{i=1}^{k-1} c_i \delta_{a_i} + c_k \delta_1. \tag{2.4}$$

Then ξ_0 is a probability measure. For $n = 0$, we denote $\gamma_0(W_x)$ by 1. We note that for $n \geqslant 1$ the moments associated with W_x are

$$\begin{aligned} \gamma_n(W_x) &\equiv x_0^2 x_1^2 x_2^2 \cdots x_{n-1}^2 \\ &= \left(\sum_{i=1}^{k-1} c_i a_i + c_k\right) \cdot \left(\frac{\sum_{i=1}^{k-1} c_i a_i^2 + c_k}{\sum_{i=1}^{k-1} c_i a_i + c_k}\right) \cdots \left(\frac{\sum_{i=1}^{k-1} c_i a_i^n + c_k}{\sum_{i=1}^{k-1} c_i a_i^{n-1} + c_k}\right) \\ &= \sum_{i=1}^{k-1} c_i a_i^n + c_k = \int s^n \, d\xi_0(s) \quad (n \geqslant 1). \end{aligned}$$

Thus, it follows that W_x is subnormal, with Berger measure

$$d\xi_0(s) = c_0 d\delta_0(s) + \sum_{i=1}^{k-1} c_i d\delta_{a_i}(s) + c_k d\delta_1(s).$$

Lemma 2.3. *For $k \geqslant 2$, $0 < a_i$ $(1 \leqslant i \leqslant k-1)$, $0 \leqslant c_0, c_1, \ldots, c_k \leqslant 1$ (with $\sum_{i=0}^k c_i = 1$), $h_0 \geqslant 1$, $k_1 \geqslant 0$, we let*

$$G(h_0, k_1, k) := \begin{pmatrix} \sum_{i=1}^{k-1} c_i a_i^{k_1+2h_0} + c_k & \sum_{i=1}^{k-1} c_i a_i^{k_1+3h_0} + c_k & \cdots & \sum_{i=1}^{k-1} c_i a_i^{k_1+(k+1)h_0} + c_k \\ \sum_{i=1}^{k-1} c_i a_i^{k_1+3h_0} + c_k & \sum_{i=1}^{k-1} c_i a_i^{k_1+4h_0} + c_k & \cdots & \sum_{i=1}^{k-1} c_i a_i^{k_1+(k+2)h_0} + c_k \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^{k-1} c_i a_i^{k_1+(k+1)h_0} + c_k & \sum_{i=1}^{k-1} c_i a_i^{k_1+(k+2)h_0} + c_k & \cdots & \sum_{i=1}^{k-1} c_i a_i^{k_1+2kh_0} + c_k \end{pmatrix}.$$

Then $G(h_0, k_1, k)$ is invertible and

$$\det G(h_0, k_1, k) = c_k \prod_{i=1}^{k-1} c_i a_i^{k_1+2h_0} \left(1 - a_i^{h_0}\right)^2 \cdot \prod_{i<j}^{k-1} \left(a_i^{h_0} - a_j^{h_0}\right)^2, \tag{2.5}$$

where $\prod_{i<j}^{k-1} (a_i^{h_0} - a_j^{h_0})^2 := 1$, if $k = 2$.

Proof. By a direct calculation using *Mathematica* [25], we have (2.5). The invertibility of $G(h_0, k_1, k)$ is clear from (2.5). $\square$

Lemma 2.4. *Under the conditions of Lemma* 2.3 *and for* $0 < a, y \leqslant 1$, *we let*

$$D(a,y) := \begin{pmatrix} y^2 & a^2y^2 & y^2 \\ a^2y^2 & a^2y^2 & a^2y^2 \\ y^2 & a^2y^2 & 1 \end{pmatrix},$$

$$F(a,y,h_0,0,k) := \begin{pmatrix} a^2y^2 & a^2y^2 & \cdots & a^2y^2 \\ a^2y^2 & a^2y^2 & \cdots & a^2y^2 \\ \sum_{i=1}^{k-1} c_i a_i^{h_0} + c_k & \sum_{i=1}^{k-1} c_i a_i^{2h_0} + c_k & \cdots & \sum_{i=1}^{k-1} c_i a_i^{kh_0} + c_k \end{pmatrix},$$

$$P(a,y,h_0,0,k) := \begin{pmatrix} D(a,y) & F(a,y,h_0,0,k) \\ F^*(a,y,h_0,0,k) & G(h_0,0,k) \end{pmatrix}.$$

Then we have

$$P(a,y,h_0,0,k) \geqslant 0 \quad \Leftrightarrow \quad y \leqslant \begin{cases} \min\{\sqrt{\frac{c_k}{a^2}}, \sqrt{\frac{c_0}{1-a^2}}\}, & \text{if } 0 < a < 1, \\ \sqrt{c_k}, & \text{if } a = 1. \end{cases} \tag{2.6}$$

Proof. By Lemma 2.3, since $D(h_0)$ is invertible, if we apply Lemma A.5 to $P(a,y,h_0,0,k)$, we have

$$P(a,y,h_0,0,k) \geqslant 0 \quad \Leftrightarrow \quad D(a,y) - W(a,y,h_0,0,k)^* G(h_0,0,k) W(a,y,h_0,0,k) \geqslant 0,$$

where

$$W(a,y,h_0,0,k) := \begin{pmatrix} \prod_{i=1}^{k-1} \frac{a^2y^2a_i^{h_0}}{(a_i^{h_0}-1)c_k} & \prod_{i=1}^{k-1} \frac{a^2y^2a_i^{h_0}}{(a_i^{h_0}-1)c_k} & g_1(h_0,k) \\ \frac{a^2y^2(a_1^{h_0}a_2^{h_0}\cdots a_{k-2}^{h_0}+\cdots+a_2^{h_0}a_3^{h_0}\cdots a_{k-1}^{h_0})}{(-1)\prod_{i=1}^{k-1}(a_i^{h_0}-1)c_k} & \frac{a^2y^2(a_1^{h_0}a_2^{h_0}\cdots a_{k-2}^{h_0}+\cdots+a_2^{h_0}a_3^{h_0}\cdots a_{k-1}^{h_0})}{(-1)\prod_{i=1}^{k-1}(a_i^{h_0}-1)c_k} & g_2(h_0,k) \\ \frac{a^2y^2(a_1^{h_0}a_2^{h_0}\cdots a_{k-3}^{h_0}+\cdots+a_3^{h_0}a_4^{h_0}\cdots a_{k-1}^{h_0})}{(-1)^2\prod_{i=1}^{k-1}(a_i^{h_0}-1)c_k} & \frac{a^2y^2(a_1^{h_0}a_2^{h_0}\cdots a_{k-3}^{h_0}+\cdots+a_3^{h_0}a_4^{h_0}\cdots a_{k-1}^{h_0})}{(-1)^2\prod_{i=1}^{k-1}(a_i^{h_0}-1)c_k} & g_3(h_0,k) \\ \vdots & \vdots & \vdots \\ (-1)^{k-1}\frac{a^2y^2(\sum_{i=1}^{k-1}a_i^{h_0})}{\prod_{i=1}^{k-1}(a_i^{h_0}-1)c_k} & (-1)^{k-1}\frac{a^2y^2(\sum_{i=1}^{k-1}a_i^{h_0})}{\prod_{i=1}^{k-1}(a_i^{h_0}-1)c_k} & g_{k-1}(h_0,k) \\ \frac{(-1)^k a^2y^2}{\prod_{i=1}^{k-1}(a_i^{h_0}-1)c_k} & \frac{(-1)^k a^2y^2}{\prod_{i=1}^{k-1}(a_i^{h_0}-1)c_k} & g_k(h_0,k) \end{pmatrix}$$

and

$$g_1(h_0,k) := 1 + \sum_{i=1}^{k-1} a_i^{-h_0}, \qquad g_2(h_0,k) := (-1)\left(\sum_{i=1}^{k-1} a_i^{-h_0} + \sum_{i<j}^{k-1} a_i^{-h_0}a_j^{-h_0}\right),$$

$$g_3(h_0,k) := (-1)^2\left(\sum_{i<j}^{k-1} a_i^{-h_0}a_j^{-h_0} + \sum_{i<j<\ell}^{k-1} a_i^{-h_0}a_j^{-h_0}a_\ell^{-h_0}\right),$$

$$g_{k-1}(h_0,k) := (-1)^{k-1}\left(a_1^{-h_0}a_2^{-h_0}\cdots a_{k-2}^{-h_0} + \cdots + a_2^{-h_0}a_3^{-h_0}\cdots a_{k-1}^{-h_0} + \prod_{i=1}^{k-1} a_i^{-h_0}\right),$$

$$g_k(h_0,k) := (-1)^k \prod_{i=1}^{k-1} a_i^{-h_0}.$$

A direct calculation shows that

$$\begin{aligned} &D(a,y) - W(a,y,h_0,0,k)^* G(h_0,0,k) W(a,y,h_0,0,k) \geqslant 0 \\ \Leftrightarrow\quad &\begin{pmatrix} y^2 - \frac{a^4y^4}{c_k} & a^2y^2 - \frac{a^4y^4}{c_k} & (1-a^2)y^2 \\ a^2y^2 - \frac{a^4y^4}{c_k} & a^2y^2 - \frac{a^4y^4}{c_k} & 0 \\ (1-a^2)y^2 & 0 & c_0 \end{pmatrix} \geqslant 0 \\ \Leftrightarrow\quad &\begin{pmatrix} y^2 - \frac{a^4y^4}{c_k} + \frac{(1-a^2)^2y^4}{c_0} & a^2y^2 - \frac{a^4y^4}{c_k} \\ a^2y^2 - \frac{a^4y^4}{c_k} & a^2y^2 - \frac{a^4y^4}{c_k} \end{pmatrix} =: \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} =: A \geqslant 0. \end{aligned}$$

To check $A \geqslant 0$, it is sufficient to check $a_{22} \geqslant 0$ and $\det A \geqslant 0$, because $(a_{22} \geqslant 0$ and $\det A \geqslant 0) \Rightarrow a_{11} \geqslant 0$. Thus, a straightforward calculation shows that $a_{22} \geqslant 0 \Leftrightarrow a^2y^2 \leqslant c_k$ and

$$\det A \geqslant 0 \quad \Leftrightarrow \quad \left(-y^2 + a^2y^2 + c_0\right)\left(c_k - a^2y^2\right) \geqslant 0.$$

Thus, it follows that

$$\begin{aligned} P(a, y, h_0, 0, k) \geqslant 0 \quad &\Leftrightarrow \quad \left\{a^2y^2 \leqslant c_{k-1} \text{ and } y^2\left(1-a^2\right) \leqslant c_0\right\} \\ &\Leftrightarrow \quad \begin{cases} \min\{\sqrt{\frac{c_k}{a^2}}, \sqrt{\frac{c_0}{1-a^2}}\}, & \text{if } 0 < a < 1, \\ \sqrt{c_k}, & \text{if } a = 1. \end{cases} \qquad \square \end{aligned}$$

For our main results, we recall that for $k \geqslant 1$, $\mathcal{G}_k = \{W_{(\alpha,\beta)} \in \mathcal{A}\colon\ W_{(\alpha,\beta)}|_{\mathcal{M}} \equiv (I \otimes S_a, U_+ \otimes I)$ and $\operatorname{card}(\operatorname{supp} \xi_0) \leqslant (k+1)\}$ (cf. see Fig. 1(ii)). We then have

Theorem 2.5. *For $k \geqslant 2$, we let $W_{(\alpha,\beta)} \equiv (T_1, T_2) \equiv \langle a, y, \xi_0 \rangle \in \mathcal{G}_k$ (where the 0-th horizontal slice $W_x \equiv \operatorname{shift}(x_0, x_1, \ldots)$ is as in (2.3) with $0 < c_0, c_1, \ldots, c_k < 1$). Then the following statements are equivalent:*

(i) $W_{(\alpha,\beta)} \in \mathfrak{H}_k$;
(ii) $W_{(\alpha,\beta)} \in \mathfrak{H}_\infty$;
(iii) *for some $h_0, \ell_0 \geqslant 1$, $W_{(\alpha,\beta)}^{(h_0,\ell_0)} \equiv (T_1^{h_0}, T_2^{\ell_0}) \in \bigoplus \mathfrak{H}_k$.*

Proof. (i) $\Rightarrow$ (ii): From Lemma A.2, we recall that a 2-variable weighted shift $W_{(\alpha,\beta)}$ is k-hyponormal if and only if

$$M_{\mathbf{k}}(k) = (\gamma_{\mathbf{k}+(m,n)+(p,q)})_{\substack{0 \leqslant n+m \leqslant k \\ 0 \leqslant p+q \leqslant k}} \geqslant 0,$$

for all $\mathbf{k} \in \mathbb{Z}_+^2$. It is straightforward to verify that $W_{(\alpha,\beta)}|_{\mathcal{M}} \cong (I \otimes S_a, U_+ \otimes I)$, so that $W_{(\alpha,\beta)}|_{\mathcal{M}}$ is subnormal. Since $W_{(\alpha,\beta)} \in \mathfrak{H}_k$, if we apply Lemma A.2(ii) at $\mathbf{k} = (k_1, 0)$ (all $k_1 \geqslant 0$), we have $M_{(k_1,0)}(k) \geqslant 0$. We note that for $h_0, \ell_0 \geqslant 1$ the moments $\gamma_{\mathbf{k}}$ ($\mathbf{k} \in \mathbb{Z}_+^2$) associated with $W_{(\alpha,\beta)}^{(h_0,\ell_0)}$ are

$$\gamma_{\mathbf{k}}\left(W_{(\alpha,\beta)}^{(h_0,\ell_0)}\right) = \begin{cases} 1, & \text{if } k_1 = 0 \text{ and } k_2 = 0, \\ \gamma_{k_1 h_0}(W_\alpha), & \text{if } k_1 \geqslant 1 \text{ and } k_2 = 0, \\ y^2, & \text{if } k_1 = 0 \text{ and } k_2 \geqslant 1, \\ a^2y^2, & \text{if } k_1 \geqslant 1 \text{ and } k_2 \geqslant 1. \end{cases} \tag{2.7}$$

Thus, by (2.7), we have that

$$W_{(\alpha,\beta)}^{(1,1)} \equiv W_{(\alpha,\beta)} \in \mathfrak{H}_k \quad \Rightarrow \quad M_{(k_1,0)}(k)(W_{(\alpha,\beta)}) \geqslant 0 \quad \Rightarrow \quad M_{(0,0)}(k)(W_{(\alpha,\beta)}) \geqslant 0.$$

From Lemma 2.4, a direct computation (i.e., interchanging rows and columns, discarding a redundant row and column in the moment matrix of $M_{(0,0)}(k)(W_{(\alpha,\beta)})$) shows that

$$\begin{aligned} M_{(0,0)}(k)(W_{(\alpha,\beta)}) \geqslant 0 \quad &\Leftrightarrow \quad P(a, y, 1, 0, k) \geqslant 0 \\ &\Leftrightarrow \quad y \leqslant \begin{cases} \min\{\sqrt{\frac{c_k}{a^2}}, \sqrt{\frac{c_0}{1-a^2}}\}, & \text{if } 0 < a < 1, \\ \sqrt{c_k}, & \text{if } a = 1. \end{cases} \end{aligned}$$

Thus, we have

$$W_{(\alpha,\beta)} \in \mathfrak{H}_k \quad \Rightarrow \quad y \leqslant \begin{cases} \min\{\sqrt{\frac{c_k}{a^2}}, \sqrt{\frac{c_0}{1-a^2}}\}, & \text{if } 0 < a < 1, \\ \sqrt{c_k}, & \text{if } a = 1. \end{cases} \tag{2.8}$$

We now characterize the subnormality of $W_{(\alpha,\beta)}$ using its parametric characterizations. Lemma A.3 will help us characterize $W_{(\alpha,\beta)} \in \mathfrak{H}_\infty$. Since $W_{(\alpha,\beta)}|_{\mathcal{M}} \equiv (I \otimes S_a, U_+ \otimes I)$ is subnormal with Berger measure $\mu_{\mathcal{M}} \equiv [(1-a^2)\delta_0 + a^2\delta_1] \times \delta_1$, we can think of $W_{(\alpha,\beta)}$ as a backward extension of $W_{(\alpha,\beta)}|_{\mathcal{M}}$ (in the t direction) and apply Lemma A.3. Note that $d(\mu_{\mathcal{M}})_{ext}(s,t) \equiv [(1-a^2)\delta_0 + a^2\delta_1] \times \delta_1$ and $\alpha_{00}^2 \|\frac{1}{t}\|_{L^1(\mu_{\mathcal{M}})} (\mu_{\mathcal{M}})_{ext}^X = y^2[(1-a^2)\delta_0 + a^2\delta_1]$. From Lemma 2.4 observe that

$$P(a,y,1,0,k)\geqslant 0 \quad\Leftrightarrow\quad y^2\big[\big(1-a^2\big)\delta_0+a^2\delta_1\big]\leqslant c_0\delta_0+\sum_{i=1}^{k-1}c_i\delta_{a_i}+c_k\delta_1$$

$$\Leftrightarrow\quad \alpha_{00}^2\left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})}(\mu_{\mathcal{M}})_{ext}^X\leqslant\xi_0,\quad \text{where } \xi_0 \text{ is the Berger measure of } W_x \text{ as in (2.3).}$$

Thus, by Lemma A.3, we have

$$W_{(\alpha,\beta)}\in\mathfrak{H}_\infty \quad\Leftrightarrow\quad P(a,y,1,0,k)\geqslant 0.$$

Therefore, by Lemma 2.4 and (2.8), it follows that $W_{(\alpha,\beta)}\in\mathfrak{H}_k\Rightarrow W_{(\alpha,\beta)}\in\mathfrak{H}_\infty$.

(ii) ⇒ (iii): Since $W_{(\alpha,\beta)}\in\mathfrak{H}_\infty$, for all $h_0,\ell_0\geqslant 1$, we have $W_{(\alpha,\beta)}^{(h_0,\ell_0)}\in\bigoplus\mathfrak{H}_\infty$ by the functional calculus. Thus we get for some $h_0,\ell_0\geqslant 1$, $W_{(\alpha,\beta)}^{(h_0,\ell_0)}\in\bigoplus\mathfrak{H}_k$.

(iii) ⇒ (i): For some $h_0,\ell_0\geqslant 1$, we suppose that $W_{(\alpha,\beta)}^{(h_0,\ell_0)}\in\bigoplus\mathfrak{H}_k$. Fixed $h_0,\ell_0\geqslant 1$, we first let $\mathcal{H}_{(m,n)}:=\bigvee_{i,j=0}^{\infty}\{e_{(h_0i+m,\ell_0j+n)}\colon h_0,\ell_0\geqslant 1\}$, for $0\leqslant m\leqslant h_0-1$ and $0\leqslant n\leqslant\ell_0-1$. Then we have

$$\ell^2\big(\mathbb{Z}_+^2\big)\equiv\bigoplus_{m=0}^{h_0-1}\bigoplus_{n=0}^{\ell_0-1}\mathcal{H}_{(m,n)}.$$

Observe that $\mathcal{H}_{(m,n)}$ reduces $T_1^{h_0}$ and $T_2^{\ell_0}$. Thus, if a 2-variable weighted shift $W_{(\alpha,\beta)}$ is given in Fig. 1(ii), then for $h_0,\ell_0\geqslant 1$, we can write

$$\begin{aligned}W_{(\alpha,\beta)}^{(h_0,\ell_0)}&\equiv\big(T_1^{h_0},T_2^{\ell_0}\big)\\&\cong\big(W_{\alpha(h_0:0)}\oplus(I\otimes S_a),T_2|_{\mathcal{H}_0}\big)\oplus\bigoplus_{i=1}^{h_0-1}\big(W_{\alpha(h_0:i)}\oplus(I\otimes U_+),T_2|_{\mathcal{H}_i}\big)\quad\big[\text{cf. see Fig. 2(i)}\big],\end{aligned}$$

where

$$W_{\alpha(h_0:i)}=\mathrm{shift}\left(\sqrt{\frac{\gamma_{(i+1)h_0}}{\gamma_{ih_0}}},\sqrt{\frac{\gamma_{(i+2)h_0}}{\gamma_{(i+1)h_0}}},\dots\right)\quad\text{and}\quad\mathcal{H}_i:=\bigoplus_{n=0}^{\ell_0-1}\mathcal{H}_{(i,n)}\quad(0\leqslant i\leqslant k-1).$$

Thus, we observe that $W_{(\alpha,\beta)}^{(h_0,\ell_0)}\in\bigoplus\mathfrak{H}_k$ is equivalent to $(W_{\alpha(h_0:0)}\oplus(I\otimes S_a),T_2|_{\mathcal{H}_0})\in\mathfrak{H}_k$ and $(W_{\alpha(h_0:i)}\oplus(I\otimes U_+),T_2|_{\mathcal{H}_i})\in\mathfrak{H}_k$, for $1\leqslant i\leqslant h_0-1$. To show $W_{(\alpha,\beta)}\in\mathfrak{H}_k$, by Lemma A.2, it is enough to show that $W_{(\alpha,\beta)}|_{\mathcal{M}}\in\mathfrak{H}_k$, $W_{(\alpha,\beta)}|_{\mathcal{N}}\in\mathfrak{H}_k$ and $M_{(0,0)}(k)(W_{(\alpha,\beta)})\geqslant 0$. Since $W_{(\alpha,\beta)}|_{\mathcal{M}}\cong(I\otimes S_a,U_+\otimes I)$ is subnormal, we need to show $W_{(\alpha,\beta)}|_{\mathcal{N}}\in\mathfrak{H}_k$ and $M_{(0,0)}(k)(W_{(\alpha,\beta)})\geqslant 0$. For $W_{(\alpha,\beta)}|_{\mathcal{N}}\in\mathfrak{H}_k$, we first want to show that for some $h_0,\ell_0\geqslant 1$

$$\big(W_{(\alpha,\beta)}|_{\mathcal{N}}\big)^{(h_0,\ell_0)}\in\bigoplus\mathfrak{H}_k\quad\Leftrightarrow\quad W_{(\alpha,\beta)}|_{\mathcal{N}}\in\mathfrak{H}_k.\tag{2.9}$$

Since $W_{(\alpha,\beta)}|_{\mathcal{M}\cap\mathcal{N}}\cong(I\otimes U_+,U_+\otimes I)$ is subnormal, to show (2.9), we need to show that

$$M_{(k_1,0)}(k)\big(W_{(\alpha,\beta)}^{(h_0,\ell_0)}\big)\geqslant 0\quad\Leftrightarrow\quad M_{(k_1,0)}(k)(W_{(\alpha,\beta)})\geqslant 0\quad\text{for } k_1\geqslant 1.$$

Since $G(h_0,k_1,k)$ is invertible, by (2.7), Fig. 2(ii) and Lemma A.5, for $k_1\geqslant 1$ and some $h_0,\ell_0\geqslant 1$, we have that

$$M_{(k_1,0)}(k)\big(W_{(\alpha,\beta)}^{(h_0,\ell_0)}\big)\geqslant 0\quad\Leftrightarrow\quad M\geqslant 0\quad\Leftrightarrow\quad a^2y^2\leqslant c_k\quad\Leftrightarrow\quad M_{(k_1,0)}(2)(W_{(\alpha,\beta)})\geqslant 0,$$

where

$$M:=\begin{pmatrix}\begin{pmatrix}a^2y^2 & a^2y^2\\ a^2y^2 & \sum_{i=1}^{k-1}c_ia_i^{h_0}+c_k\end{pmatrix} & \begin{pmatrix}a^2y^2 & \cdots & a^2y^2\\ \sum_{i=1}^{k-1}c_ia_i^{k_1+h_0}+c_k & \cdots & \sum_{i=1}^{k-1}c_ia_i^{k_1+kh_0}+c_k\end{pmatrix}\\ \begin{pmatrix}a^2y^2 & \sum_{i=1}^{k-1}c_ia_i^{k_1+h_0}+c_k\\ \vdots & \vdots\\ a^2y^2 & \sum_{i=1}^{k-1}c_ia_i^{k_1+kh_0}+c_k\end{pmatrix} & G(h_0,k_1,k)\end{pmatrix}.$$

Thus, we have

$$\big(W_{(\alpha,\beta)}|_{\mathcal{N}}\big)^{(h_0,\ell_0)}\in\bigoplus\mathfrak{H}_k\quad\Leftrightarrow\quad W_{(\alpha,\beta)}|_{\mathcal{N}}\in\mathfrak{H}_k.$$

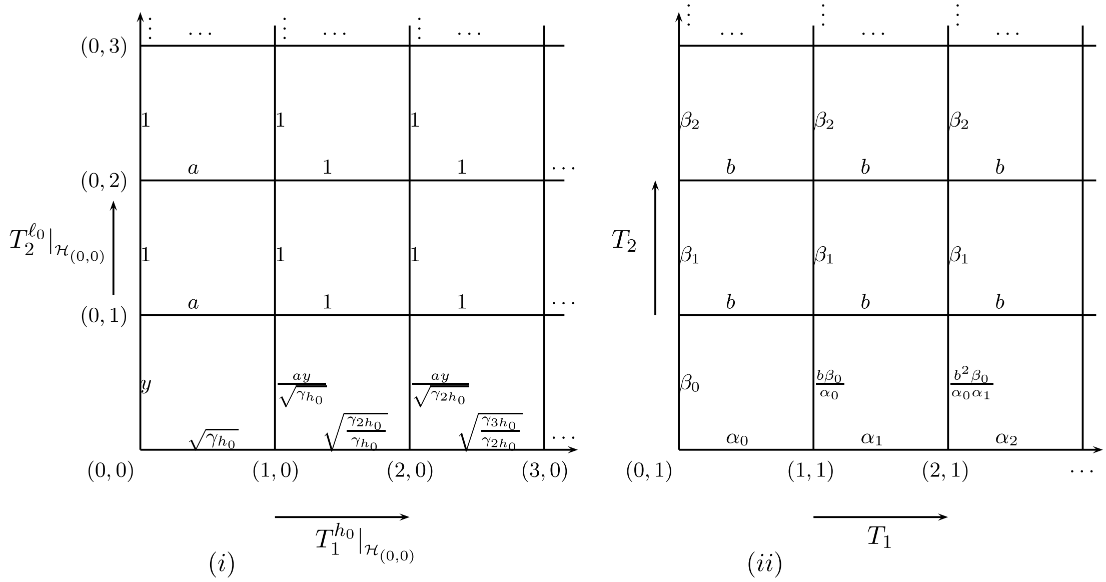

Fig. 2. Weight diagram of the 2-variable weighted shift $W^{(h_0,\ell_0)}_{(\alpha,\beta)}$ in Theorem 2.5 and weight diagram of the 2-variable weighted shift in Lemma A.4, respectively.

For $M_{(0,0)}(k)(W_{(\alpha,\beta)}) \geqslant 0$, we note that

$$W^{(h_0,\ell_0)}_{(\alpha,\beta)} \in \bigoplus \mathfrak{H}_k \quad \Rightarrow \quad \big(W_{\alpha(h_0:0)} \oplus (I \otimes S_a), T_2|_{\mathcal{H}_0}\big) \in \mathfrak{H}_k.$$

We also observe that

$$\big(W_{\alpha(h_0:0)} \oplus (I \otimes S_a), T_2|_{\mathcal{H}_0}\big) \cong \bigoplus_{n=0}^{\ell_0-1} \big(W_{\alpha(h_0:0)} \oplus (I \otimes S_a), T_2^{\ell_0}\big|_{\mathcal{H}_{(0,n)}}\big)$$

and

$$\bigoplus_{n=0}^{\ell_0-1} \big(W_{\alpha(h_0:0)} \oplus (I \otimes S_a), T_2^{\ell_0}\big|_{\mathcal{H}_{(0,n)}}\big) \cong \big(W_{\alpha(h_0:0)} \oplus (I \otimes S_a), T_2^{\ell_0}\big|_{\mathcal{H}_{(0,0)}}\big) \oplus \bigoplus_{n=0}^{\ell_0-1} (I \otimes S_a, U_+ \otimes I).$$

Note that the second summand is clearly subnormal; thus, for $h_0, \ell_0 \geqslant 1$, the k-hyponormality of $(W_{\alpha(h_0:0)} \oplus (I \otimes S_a), T_2|_{\mathcal{H}_0})$ is equivalent to the k-hyponormality of the first summand, $(W_{\alpha(h_0:0)} \oplus (I \otimes S_a), T_2^{\ell_0}|_{\mathcal{H}_{(0,0)}})$. Observe also that

$$\big(W_{\alpha(h_0:0)} \oplus (I \otimes S_a), T_2^{\ell_0}\big|_{\mathcal{H}_{(0,0)}}\big) \cong \big(W_{\alpha(h_0:0)} \oplus (I \otimes S_a), T_2|_{\mathcal{H}_{(0,0)}}\big).$$

Thus we have

$$\big(W_{\alpha(h_0:0)} \oplus (I \otimes S_a), T_2|_{\mathcal{H}_0}\big) \in \mathfrak{H}_k \quad \Leftrightarrow \quad \big(W_{\alpha(h_0:0)} \oplus (I \otimes S_a), T_2|_{\mathcal{H}_{(0,0)}}\big) \in \mathfrak{H}_k.$$

To check the k-hyponormality of $(W_{\alpha(h_0:0)} \oplus (I \otimes S_a), T_2|_{\mathcal{H}_{(0,0)}})$, we observe that it suffices to apply Lemma A.2(ii) at $\mathbf{k} = (0,0)$. From Lemma 2.4 and (2.7), after we apply Lemma A.2 to $(0,0)$ of $(W_{\alpha(h_0:0)} \oplus (I \otimes U_+), T_2|_{\mathcal{H}_0})$, we have

$$M_{(0,0)}(k)\big(W_{\alpha(h_0:0)} \oplus (I \otimes U_+), T_2|_{\mathcal{H}_0}\big) \geqslant 0 \quad \Leftrightarrow \quad P(a, y, h_0, 0, k) \geqslant 0 \quad \Leftrightarrow \quad y^2 \leqslant c_2,$$

where $P(a, y, h_0, 0, k)$ is as in Lemma 2.4. Thus we get

$$W^{(h_0,\ell_0)}_{(\alpha,\beta)} \in \bigoplus \mathfrak{H}_k \quad \Rightarrow \quad y^2 \leqslant c_k \quad \Leftrightarrow \quad M_{(0,0)}(k)(W_{(\alpha,\beta)}) \geqslant 0. \tag{2.10}$$

Therefore, we have for some $h_0, \ell_0 \geqslant 1$,

$$W^{(h_0,\ell_0)}_{(\alpha,\beta)} \in \bigoplus \mathfrak{H}_k \quad \Rightarrow \quad W_{(\alpha,\beta)} \in \mathfrak{H}_k$$

and our proof is now complete. □

Remark 2.6.

(i) We note that $\mathrm{card}(\mathrm{supp}\,\xi_0) = (k+1)$, where ξ_0 is as in Theorem 2.5.
(ii) In Theorem 2.5, we show that for given $k \geqslant 2$, there exists a 2-variable weighted shift $W_{(\alpha,\beta)} \in \mathcal{G}_k \subsetneq \mathcal{A}$ for which some $h_0, \ell_0 \geqslant 1$

$$W_{(\alpha,\beta)}^{(h_0,\ell_0)} \in \bigoplus \mathfrak{H}_k \quad \Leftrightarrow \quad W_{(\alpha,\beta)} \in \mathfrak{H}_\infty.$$

(iii) By Theorem 2.5, we note that there exists a 2-variable weighted shift $W_{(\alpha,\beta)} \in \mathcal{G}_2$ in $\mathcal{A}$ for which some $h_0, \ell_0 \geqslant 1$

$$W_{(\alpha,\beta)}^{(h_0,\ell_0)} \in \bigoplus \mathfrak{H}_k \quad \Leftrightarrow \quad W_{(\alpha,\beta)} \in \mathfrak{H}_\infty.$$

Thus, Theorem 2.5 gives an answer for Problem 2.1 and more.

Observe that if $W_{(\alpha,\beta)} \equiv \langle a, y, \xi_0\rangle \in \mathcal{G}_k$ $(k \geqslant 2)$, then by (2.4) the measure ξ_0 can be completely determined by the $(2k-1)$ parameters $\{a_i\}_{i=1}^{k-1}$ and $\{c_i\}_{i=0}^{k}$. Thus we can also denote a 2-variable weighted shift $W_{(\alpha,\beta)} \equiv \langle a, y, \xi_0\rangle \in \mathcal{G}_k$ by $\langle a, y, \{a_i\}_{i=1}^{k-1}, \{c_i\}_{i=0}^{k}\rangle$ (cf. see Fig. 1(ii)). We now assume $ay \leqslant \sum_{i=1}^{k-1} c_i a_i^h + c_k$ (all $h \geqslant 1$), because we need to ensure that $W_{(\alpha,\beta)} \in \mathfrak{H}_0$. We now obtain a canonical representation for the powers $\langle a, y, \{a_i\}_{i=1}^{k-1}, \{c_i\}_{i=0}^{k}\rangle^{(h,\ell)}$ as an orthogonal direct sum of 2-variable weighted shifts in $\mathcal{G}_k$. In what follows, we abbreviate the orthogonal direct sums of h copies of a shift $\langle a, y, \{a_i\}_{i=1}^{k-1}, \{c_i\}_{i=0}^{k}\rangle$ by $h \cdot \langle a, y, \{a_i\}_{i=1}^{k-1}, \{c_i\}_{i=0}^{k}\rangle$. Then we have

Proposition 2.7. *We let $\langle a, y, \{a_i\}_{i=1}^{k-1}, \{c_i\}_{i=0}^{k}\rangle \equiv \langle a, y, \{a_i\}_{i=1}^{k-1}, c_0, \{c_i\}_{i=1}^{k-1}, c_k\rangle \in \mathcal{G}_k$. Then for $h, \ell \geqslant 1$, we have*

$$\begin{aligned}
&\big\langle a, y, \{a_i\}_{i=1}^{k-1}, c_0, \{c_i\}_{i=1}^{k-1}, c_k\big\rangle^{(h,\ell)} \\
&\quad \cong \big\langle a, y, \big\{a_i^{\frac{1}{h}}\big\}_{i=1}^{k-1}, c_0, \{c_i\}_{i=1}^{k-1}, c_k\big\rangle \\
&\qquad \oplus \left\langle 1, \frac{ay}{\sqrt{\sum_{i=1}^{k-1} c_i a_i^{mh} + c_k}}, \big\{a_i^{\frac{1}{h}}\big\}_{i=1}^{k-1}, 0, \left\{\frac{a_i^{\frac{1}{h}} c_i}{\sqrt{\sum_{i=1}^{k-1} c_i a_i^{m} + c_k}}\right\}_{i=1}^{k-1}, \frac{c_k}{\sqrt{\sum_{i=1}^{k-1} c_i a_i^{m} + c_k}}\right\rangle \\
&\qquad \oplus (\ell-1)\cdot\big\langle a, 1, 0, 1-a^2, 0, a^2\big\rangle \oplus (h-1)(\ell-1)\cdot\langle 1, 1, 0, 0, 0, 1\rangle,
\end{aligned}$$

so that the class $\mathcal{G}_k$ is invariant under all powers.

Proof. We recall that we decompose the space $\ell^2(\mathbb{Z}_+^2)$ as the orthogonal direct sum of $h\ell$ subspaces $\mathcal{H}_{(m,n)}$, each isometrically isomorphic to $\ell^2(\mathbb{Z}_+^2)$, namely $\mathcal{H}_{(m,n)} := \bigvee_{i,j=0}^{\infty}\{e_{(hi+m,\ell j+n)}\}$ $(0 \leqslant m \leqslant h-1,\ 0 \leqslant n \leqslant \ell-1)$. This particular decomposition allows us to write the power

$$\big\langle a, y, \{a_i\}_{i=1}^{k-1}, c_0, \{c_i\}_{i=1}^{k-1}, c_k\big\rangle^{(h,\ell)}$$

as the orthogonal direct sum

$$\bigoplus_{0\leqslant m\leqslant h-1,\,0\leqslant n\leqslant \ell-1}^{(h,\ell)} \big\langle a, y, \{a_i\}_{i=1}^{k-1}, c_0, \{c_i\}_{i=1}^{k-1}, c_k\big\rangle\big|_{\mathcal{H}_{(m,n)}}.$$

From (2.2), we will now identify each of the summands $\langle a, y, b, c_0, c_1, c_2\rangle^{(h,\ell)}|_{\mathcal{H}_{(m,n)}}$ $(0 \leqslant m \leqslant h-1,\ 0 \leqslant n \leqslant \ell-1)$.

Case 1: $(m=0,\ n=0)$ Direct inspection of the weight families α and β shows that

$$\big\langle a, y, \{a_i\}_{i=1}^{k-1}, c_0, \{c_i\}_{i=1}^{k-1}, c_k\big\rangle^{(h,\ell)} e_{(hi,\ell j)} = \big\langle a, y, \big\{a_i^{\frac{1}{h}}\big\}_{i=1}^{k-1}, c_0, \{c_i\}_{i=1}^{k-1}, c_k\big\rangle e_{(hi,\ell j)},$$

and therefore

$$\big\langle a, y, \{a_i\}_{i=1}^{k-1}, c_0, \{c_i\}_{i=1}^{k-1}, c_k\big\rangle^{(h,\ell)}\big|_{\mathcal{H}_{(0,0)}} \cong \big\langle a, y, \big\{a_i^{\frac{1}{h}}\big\}_{i=1}^{k-1}, c_0, \{c_i\}_{i=1}^{k-1}, c_k\big\rangle.$$

Case 2: ($m>0$, $n=0$) In this case the generic basis vector of $\mathcal{H}_{(m,0)}$ is $e_{(hi+m,\ell j)}$, so that

$$\begin{aligned}&\big\langle a, y, \{a_i\}_{i=1}^{k-1}, c_0, \{c_i\}_{i=1}^{k-1}, c_k\big\rangle^{(h,\ell)} e_{(hi+m,\ell j)}\\ &=\left\langle 1, \frac{ay}{\sqrt{\sum_{i=1}^{k-1} c_i a_i^{mh} + c_k}}, \big\{a_i^{\frac{1}{h}}\big\}_{i=1}^{k-1}, 0, \left\{\frac{a_i^{\frac{1}{h}} c_i}{\sqrt{\sum_{i=1}^{k-1} c_i a_i^{m} + c_k}}\right\}_{i=1}^{k-1}, \frac{c_k}{\sqrt{\sum_{i=1}^{k-1} c_i a_i^{m} + c_k}}\right\rangle e_{(hi+m,\ell j)}.\end{aligned}$$

It follows that

$$\begin{aligned}&\big\langle a, y, \{a_i\}_{i=1}^{k-1}, c_0, \{c_i\}_{i=1}^{k-1}, c_k\big\rangle^{(h,\ell)}\big|_{\mathcal{H}_{(m,0)}}\\ &\cong\left\langle 1, \frac{ay}{\sqrt{\sum_{i=1}^{k-1} c_i a_i^{mh} + c_k}}, \big\{a_i^{\frac{1}{h}}\big\}_{i=1}^{k-1}, 0, \left\{\frac{a_i^{\frac{1}{h}} c_i}{\sqrt{\sum_{i=1}^{k-1} c_i a_i^{m} + c_k}}\right\}_{i=1}^{k-1}, \frac{c_k}{\sqrt{\sum_{i=1}^{k-1} c_i a_i^{m} + c_k}}\right\rangle.\end{aligned}$$

Case 3: ($m=0$, $n>0$) In this case the generic basis vector of $\mathcal{H}_{(0,n)}$ is $e_{(hi,\ell j+n)}$, and therefore

$$\big\langle a, y, \{a_i\}_{i=1}^{k-1}, c_0, \{c_i\}_{i=1}^{k-1}, c_k\big\rangle^{(h,\ell)} e_{(hi,\ell j+n)} = \big\langle a, 1, 0, 1-a^2, 0, a^2\big\rangle e_{(hi,\ell j+n)}.$$

It follows that $\langle a, y, \{a_i\}_{i=1}^{k-1}, c_0, \{c_i\}_{i=1}^{k-1}, c_k\rangle^{(h,\ell)}|_{\mathcal{H}_{(0,n)}} \cong \langle a, 1, 0, 1-a^2, 0, a^2\rangle$.

Case 4: ($m>0$, $n>0$) Since $\mathcal{H}_{(m,n)} \subseteq \mathcal{M}\cap\mathcal{N}$, and the core of $W_{(\alpha,\beta)}$ is trivial, it is clear that all relevant weights are equal to 1, so

$$\big\langle a, y, \{a_i\}_{i=1}^{k-1}, c_0, \{c_i\}_{i=1}^{k-1}, c_k\big\rangle^{(h,\ell)} e_{(hi+m,\ell j+n)} = \langle 1, 1, 0, 0, 0, 1\rangle e_{(hi+m,\ell j+n)},$$

and therefore $\langle a, y, \{a_i\}_{i=1}^{k-1}, c_0, \{c_i\}_{i=1}^{k-1}, c_k\rangle^{(h,\ell)}|_{\mathcal{H}_{(m,n)}} \cong \langle 1, 1, 0, 0, 0, 1\rangle$.

Therefore, our proof is now complete. □

Corollary 2.8. *For $\langle a, y, \{a_i\}_{i=1}^{k-1}, \{c_i\}_{i=0}^{k}\rangle \in \mathcal{G}_k$ ($k \geqslant 2$), the following statements are equivalent*:

(i) *for some $h_0, \ell_0 \geqslant 1$, $\langle a, y, \{a_i\}_{i=1}^{k-1}, \{c_i\}_{i=0}^{k}\rangle^{(h_0,\ell_0)} \in \bigoplus \mathfrak{H}_k$;*
(ii) *for all $h, \ell \geqslant 1$, $\langle a, y, \{a_i\}_{i=1}^{k-1}, \{c_i\}_{i=0}^{k}\rangle^{(h,\ell)} \in \bigoplus \mathfrak{H}_k$;*
(iii) $\langle a, y, \{a_i\}_{i=1}^{k-1}, \{c_i\}_{i=0}^{k}\rangle \in \mathfrak{H}_k$;
(iv) *for some $h_0, \ell_0 \geqslant 1$, $\langle a, y, \{a_i\}_{i=1}^{k-1}, \{c_i\}_{i=0}^{k}\rangle^{(h_0,\ell_0)} \in \bigoplus \mathfrak{H}_\infty$;*
(v) *for all $h, \ell \geqslant 1$, $\langle a, y, \{a_i\}_{i=1}^{k-1}, \{c_i\}_{i=0}^{k}\rangle^{(h,\ell)} \in \bigoplus \mathfrak{H}_\infty$;*
(vi) $\langle a, y, \{a_i\}_{i=1}^{k-1}, \{c_i\}_{i=0}^{k}\rangle \in \mathfrak{H}_\infty$.

Proof. This is straightforward from Theorem 2.5, Proposition 2.7 and the functional calculus. □

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Appendix A

For the reader's convenience, in this section we gather several well-known auxiliary results which are needed for the proofs of the main results in this article. First, to detect hyponormality for 2-variable weighted shifts we use a simple criterion involving a base point $\mathbf{k}$ in $\mathbb{Z}_+^2$ and its five neighboring points in $\mathbf{k}+\mathbb{Z}_+^2$ at path distance at most 2.

Lemma A.1 *(Six-point Test). (See [3, Theorem 6.1].) Let $W_{(\alpha,\beta)} \equiv (T_1, T_2)$ be a 2-variable weighted shift, with weight sequences α and β. Then*

$$\begin{aligned}&\big[W_{(\alpha,\beta)}^*, W_{(\alpha,\beta)}\big] \geqslant 0\\ &\Leftrightarrow\quad H(k_1,k_2)(1) := \begin{pmatrix} \alpha_{\mathbf{k}+\varepsilon_1}^2 - \alpha_{\mathbf{k}}^2 & \alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}+\varepsilon_1} - \alpha_{\mathbf{k}}\beta_{\mathbf{k}} \\ \alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}+\varepsilon_1} - \alpha_{\mathbf{k}}\beta_{\mathbf{k}} & \beta_{\mathbf{k}+\varepsilon_2}^2 - \beta_{\mathbf{k}}^2 \end{pmatrix} \geqslant 0 \quad \big(\textit{for all } \mathbf{k} \in \mathbb{Z}_+^2\big).\end{aligned}$$

Next, we present an analogous criterion to detect the k-hyponormality of 2-variable weighted shifts.

Lemma A.2. *(See [5, Theorem 2.4].) Let $W_{(\alpha,\beta)}$ be a 2-variable weighted shift with weight sequence α and β. The following statements are equivalent*:

(i) $W_{(\alpha,\beta)}$ *is k-hyponormal*;
(ii) $M_{\mathbf{k}}(k) := (\gamma_{\mathbf{k}+(n,m)+(p,q)})_{\substack{0\leqslant n+m\leqslant k\\ 0\leqslant p+q\leqslant k}} \geqslant 0$ *for all* $\mathbf{k} \in \mathbb{Z}_+^2$.

To check subnormality of 2-variable weighted shifts, we introduce some definitions.

(i) Let μ and ν be two positive measures on $\mathbb{R}_+$. We say that $\mu \leqslant \nu$ on $X := \mathbb{R}_+$, if $\mu(E) \leqslant \nu(E)$ for all Borel subset $E \subseteq \mathbb{R}_+$; equivalently, $\mu \leqslant \nu$ if and only if $\int f\,d\mu \leqslant \int f\,d\nu$ for all $f \in C(X)$ such that $f \geqslant 0$ on $\mathbb{R}_+$.
(ii) Let μ be a probability measure on $X \times Y$, and assume that $\frac{1}{t} \in L^1(\mu)$. The *extremal measure* μ_{ext} (which is also a probability measure) on $X \times Y$ is given by $d\mu_{ext}(s,t) := (1-\delta_0(t))\frac{1}{t\|\frac{1}{t}\|_{L^1(\mu)}}d\mu(s,t)$.
(iii) Given a measure μ on $X \times Y$, the *marginal measure* μ^X is given by $\mu^X := \mu \circ \pi_X^{-1}$, where $\pi_X : X \times Y \to X$ is the canonical projection onto X. Thus $\mu^X(E) = \mu(E \times Y)$, for every $E \subseteq X$.

Then we have:

Lemma A.3 *(Subnormal backward extension). (See [13, Proposition 3.10].) Let $W_{(\alpha,\beta)}$ be a 2-variable weighted shift, and assume that $W_{(\alpha,\beta)}|_{\mathcal{M}}$ is subnormal with associated measure $\mu_{\mathcal{M}}$ and that $W_0 := \mathrm{shift}(\alpha_{00}, \alpha_{10}, \ldots)$ is subnormal with associated measure ξ_0. Then $W_{(\alpha,\beta)}$ is subnormal if and only if*

(i) $\frac{1}{t} \in L^1(\mu_{\mathcal{M}})$;
(ii) $\beta_{00}^2 \leqslant (\|\frac{1}{t}\|_{L^1(\mu_{\mathcal{M}})})^{-1}$;
(iii) $\beta_{00}^2\|\frac{1}{t}\|_{L^1(\mu_{\mathcal{M}})}(\mu_{\mathcal{M}})_{ext}^X \leqslant \xi_0$.

Moreover, if $\beta_{00}^2\|\frac{1}{t}\|_{L^1(\mu_{\mathcal{M}})} = 1$, then $(\mu_{\mathcal{M}})_{ext}^X = \xi_0$. In the case when $W_{(\alpha,\beta)}$ is subnormal, the Berger measure μ of $W_{(\alpha,\beta)}$ is given by

$$d\mu(s,t) = \beta_{00}^2\left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} d(\mu_{\mathcal{M}})_{ext}(s,t) + \left(d\xi_0(s) - \beta_{00}^2\left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} d(\mu_{\mathcal{M}})_{ext}^X(s)\right)d\delta_0(t).$$

Lemma A.4. *(See [27, Theorem 2.8].) Let $W_{(\alpha,\beta)} \in \mathfrak{H}_0$ be a 2-variable weighted shift whose weight diagram is given in Fig. 2(ii), so that $W_{(\alpha,\beta)}|_{\mathcal{M}} \cong (I \otimes \mathrm{shift}(\beta_1, \beta_2, \ldots), U_+ \otimes bI)$. Assume that $\|W_\alpha\| = b > 0$, where $W_\alpha \equiv \mathrm{shift}(\alpha_0, \alpha_1, \alpha_2, \ldots)$. Then $W_{(\alpha,\beta)} \in \mathfrak{H}_1 \Leftrightarrow W_{(\alpha,\beta)} \in \mathfrak{H}_\infty \Leftrightarrow$ the Berger measure μ_α of W_α has an atom at b^2.*

Lemma A.5. *(Cf. [4, Proposition 2.2], [16,23].) Let $M \equiv \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ be a 2×2 operator matrix, where A and C are square matrices and B is a rectangular matrix. Then*

$$M \geqslant 0 \quad \Leftrightarrow \quad \text{there exists } W \text{ such that } \begin{cases} A \geqslant 0, \\ B = AW, \\ C \geqslant W^*AW. \end{cases}$$

References

[1] A. Athavale, On joint hyponormality of operators, Proc. Amer. Math. Soc. 103 (1988) 417–423.
[2] J. Conway, The Theory of Subnormal Operators, Math. Surveys Monogr., vol. 36, Amer. Math. Soc., Providence, 1991.
[3] R. Curto, Joint hyponormality: A bridge between hyponormality and subnormality, Proc. Sympos. Pure Math. 51 (1990) 69–91.
[4] R. Curto, L. Fialkow, Solution of the Truncated Complex Moment Problem with Flat Data, Mem. Amer. Math. Soc., vol. 568, Amer. Math. Soc., Providence, 1996.
[5] R. Curto, S.H. Lee, J. Yoon, k-Hyponormality of multivariable weighted shifts, J. Funct. Anal. 229 (2005) 462–480.
[6] R. Curto, S.H. Lee, J. Yoon, Hyponormality and subnormality for powers of commuting pairs of subnormal operators, J. Funct. Anal. 245 (2007) 390–412.
[7] R. Curto, S.H. Lee, J. Yoon, Reconstruction of the Berger measure when the core is of tensor form, in: Actas del XVI Coloquio Latinoamericano de Álgebra, Bibl. Rev. Mat. Iberoamericana (2007) 317–331.
[8] R. Curto, S.H. Lee, J. Yoon, Which 2-hyponormal 2-variable weighted shifts are subnormal?, Linear Algebra Appl. 429 (2008) 2227–2238.
[9] R. Curto, S.H. Lee, J. Yoon, New approach to the 2-variable subnormal completion problem, J. Math. Anal. Appl. 370 (2010) 270–283.
[10] R. Curto, P. Muhly, J. Xia, Hyponormal pairs of commuting operators, Oper. Theory Adv. Appl. 35 (1988) 1–22.
[11] R. Curto, S. Park, k-Hyponormality of powers of weighted shifts, Proc. Amer. Math. Soc. 131 (2002) 2762–2769.
[12] R. Curto, M. Putinar, Nearly subnormal operators and moments problems, J. Funct. Anal. 115 (1993) 480–497.
[13] R. Curto, J. Yoon, Jointly hyponormal pairs of subnormal operators need not be jointly subnormal, Trans. Amer. Math. Soc. 358 (2006) 5139–5159.
[14] R. Curto, J. Yoon, Disintegration-of-measure techniques for multivariable weighted shifts, Proc. London Math. Soc. 93 (2006) 381–402.

[15] R. Curto, J. Yoon, Propagation phenomena for hyponormal 2-variable weighted shifts, J. Operator Theory 58 (1) (2007) 101–130.
[16] R. Curto, J. Yoon, Spectral picture of 2-variable weighted shifts, C. R. Acad. Sci. Paris 343 (2006) 579–584.
[17] R. Curto, J. Yoon, When is hyponormality for 2-variable weighted shifts invariant under powers?, Indiana Univ. Math. J., submitted for publication.
[18] R. Gellar, L.J. Wallen, Subnormal weighted shifts and the Halmos–Bram criterion, Proc. Japan Acad. 46 (1970) 375–378.
[19] P.R. Halmos, A Hilbert Space Problem Book, second ed., Grad. Texts in Math., vol. 19, Springer-Verlag, New York/Berlin, 1982.
[20] N.P. Jewell, A.R. Lubin, Commuting weighted shifts and analytic function theory in several variables, J. Operator Theory 1 (1979) 207–223.
[21] A.R. Lubin, Weighted shifts and commuting normal extension, J. Austral. Math. Soc. Ser. A 27 (1) (1979) 17–26.
[22] A.L. Shields, Weighted shift operators and analytic function theory, Math. Surveys 13 (1974) 49–128.
[23] Ju.L. Smul'jan, An operator Hellinger integral, Mat. Sb. (N.S.) 49 (1959) 381–430 (in Russian).
[24] J. Stampfli, Which weighted shifts are subnormal?, Pacific J. Math. 17 (1966) 367–379.
[25] Wolfram Research, Inc., Mathematica, Version 4.2, Wolfram Research, Inc., Champaign, IL, 2002.
[26] J. Yoon, Disintegration of measures and contractive 2-variable weighted shifts, Integral Equations Operator Theory 59 (2007) 281–298.
[27] J. Yoon, Schur product techniques for commuting multivariable weighted shifts, J. Math. Anal. Appl. 333 (2007) 626–641.